Parametrization and Regularization of a 3-D Helicopter Electromagnetic Data Inversion

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Motivation

Helicopter electromagnetic (HEM) measurements make it feasible to survey large areas in a relatively short time. For the resulting data sets, laterally constrained one-dimensional inversion schemes are still state of the art and are used to quickly obtain a first model for the whole survey area. However, this will fail for regions with a significant three-dimensional conductivity structure. Consequently, these regions are identified and extracted together with the corresponding subsets of measurement data. We have developed a scheme capable of performing a full three-dimensional inversion on these subsets while respecting constraints imposed by the surrounding one-dimensional model.

Two important aspects, that are highlighted here, are the parametrization of the model domain and the regularization applied to this parametrization. Both are required to deal with the inherently under-determined problem (resulting from the employed data acquisition method) and to link the model inverted for to the given surrounding model.

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Forward Problem

- Secondary field approach for Helmholtz equation:
  - Multiple left-hand sides corresponding to measurements at different frequencies
  - Many right-hand sides corresponding to TX/RX positions along flight profiles

$$\nabla\times\nabla\times \boldsymbol{E}_{\text{sec}} + \mathrm{i}\omega\mu_0\boldsymbol{\sigma}_{\text{tot}}\boldsymbol{E}_{\text{sec}} = -\mathrm{i}\omega\mu_0\left(\boldsymbol{\sigma}_{\text{tot}} - \boldsymbol{\sigma}_{\text{pri}}\right)\boldsymbol{E}_{\text{pri}} \quad \text{in } \Omega$$
$$\boldsymbol{n}\times\boldsymbol{E}_{\text{sec}} = 0 \quad \text{on } \partial\Omega$$

- Linear system of equations after finite difference or finite element discretization:

$$\boldsymbol{A}(\boldsymbol{\sigma}_{\text{tot}})\,\boldsymbol{u}_{\text{sec}} = \boldsymbol{b}$$

$$\boldsymbol{A}(\boldsymbol{\sigma}) = \boldsymbol{K} + \mathrm{i}\omega\mu_0\boldsymbol{M}(\boldsymbol{\sigma})$$
$$\boldsymbol{b} = -\boldsymbol{A}\left(\boldsymbol{\sigma}_{\text{tot}} - \boldsymbol{\sigma}_{\text{pri}}\right)\boldsymbol{u}_{\text{pri}}$$

- $\boldsymbol{E}_{\text{pri}}$, $\boldsymbol{u}_{\text{pri}}$: Electric field of a vertical magnetic dipole in air above a layered subsurface

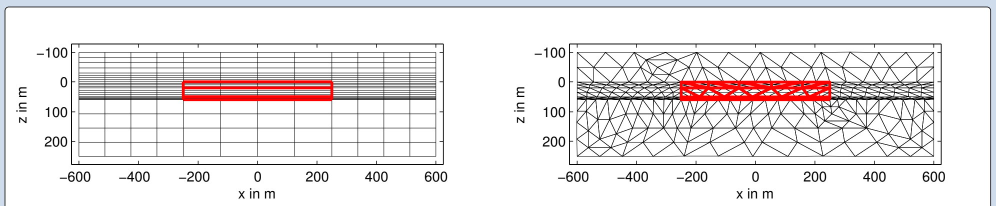
Mapping from Inverse Grid to Forward Grid

- Model parameter $\boldsymbol{m}$ lives on a subset of coarse version of forward modeling grid
- Mapping of model parameters $\boldsymbol{m}$ (and corresponding conductivities $\boldsymbol{\sigma}$):

$$\boldsymbol{m}_{\text{tot}} = (\boldsymbol{I} - \boldsymbol{E}\boldsymbol{E}^\top)\boldsymbol{m}_{\text{pri}} + \boldsymbol{E}\boldsymbol{m}$$
$$\boldsymbol{\sigma}_{\text{tot}} = \exp \boldsymbol{m}_{\text{tot}}$$

$\boldsymbol{E}$	… expansion matrix
$\boldsymbol{m}$	… model parameter (inverse grid, active region)
$\boldsymbol{m}_{\text{pri}}$	… model parameter (forward grid, background)
$\boldsymbol{m}_{\text{tot}}$	… model parameter (forward grid, total)

- Illustration of model parameter mapping:



Cross sections of an FD (left) and FE (right) forward modeling grid along the xz-plane for a simple model. Highlighted in red is the active region with the model parameter cells that each span multiple cells of the forward modeling grid. Both grids were simplified to better illustrate the idea of mapping between forward and inverse grid.

Inverse Problem

- Minimization problem for objective function Φ with Tikhonov-type regularization:

$$\Phi(\boldsymbol{m}) = \underbrace{\frac{1}{2}\|\boldsymbol{d}^{\text{obs}} - \boldsymbol{d}(\boldsymbol{m})\|_2^2}_{\text{data misfit}} + \underbrace{\frac{\lambda}{2}\|\boldsymbol{W}(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})\|_2^2}_{\text{model misfit}} \rightarrow \min_{\boldsymbol{m}}$$

with $\boldsymbol{d}(\boldsymbol{m}) = \boldsymbol{Q}(\boldsymbol{A}(\boldsymbol{m})^{-1}\boldsymbol{b} + \boldsymbol{u}_{\text{pri}})$

$\boldsymbol{d}^{\text{obs}}$	… observed data (field data)
$\boldsymbol{Q}$	… measurement operator
λ	… regularization parameter
$\boldsymbol{W}$	… weighting matrix (regularization)
$\boldsymbol{m}_{\text{ref}}$	… reference model for regularization

- Matrix $\boldsymbol{W}$ is any (weighted) combination of the available weightings:
  - Marquardt-Levenberg
  - Gradients and second order derivatives on tensor product grid (finite differences)
  - Gradients (using Raviart-Thomas elements) on unstructured grid (finite elements)
- Additional care needs to be taken to get coupling between active and inactive region right

Gauss-Newton Method

- Linearization and reformulation as normal equation yield:

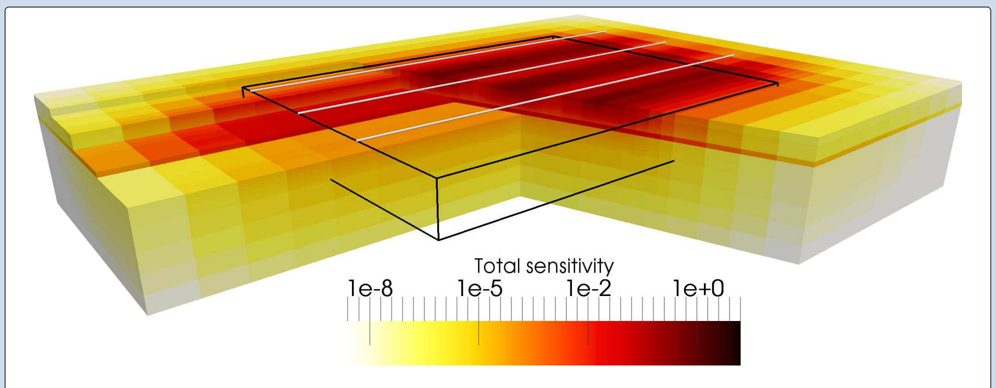
$$\left(\boldsymbol{J}^\top\boldsymbol{J} + \lambda\boldsymbol{W}^\top\boldsymbol{W}\right)\Delta\boldsymbol{m} = \boldsymbol{J}^\top\left(\boldsymbol{d}^{\text{obs}} - \boldsymbol{d}(\boldsymbol{m})\right) + \lambda\boldsymbol{W}^\top\boldsymbol{W}(\boldsymbol{m} - \boldsymbol{m}_{\text{ref}})$$

$\boldsymbol{m}$	… current model parameter
$\Delta\boldsymbol{m}$	… model parameter update
$\boldsymbol{J}$	… sensitivity matrix

- Solved for $\Delta\boldsymbol{m}$ using CG (normal equation) or LSQR (linear least squares)

Sensitivity and Data Sampling in HEM

- Measurement data is collected along profiles
  - Data density in direction of flight is significantly higher than perpendicular to it
  - Regularization needed to alleviate this problem
- Influence of typical data distribution on cumulative sensitivity:



Subset of an FD grid with cumulative sensitivities over all TX/RX positions and all frequencies for three flight profiles above a layered half-space. The active region underneath the flight profiles is hinted at by the black outline.

Regularization for Finite Differences

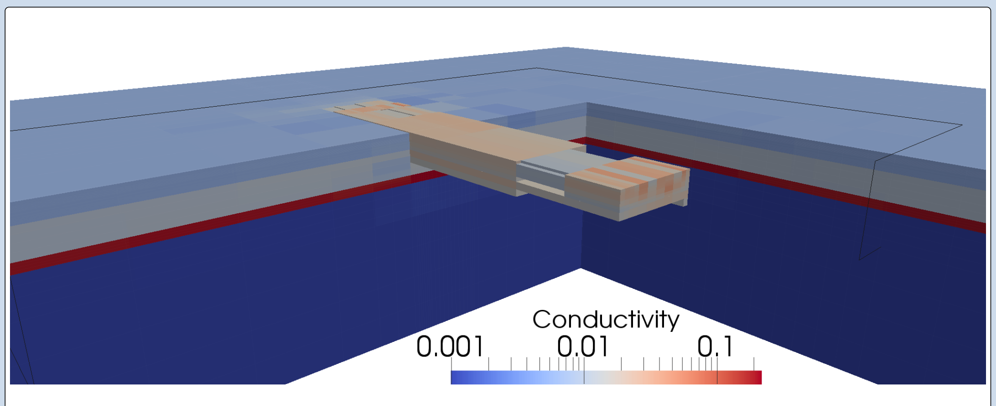
- Gradients with direction-dependent weighting:
  - Discrete gradients on Yee cell with separate FD stencils per partial derivative: $\boldsymbol{G}_x, \boldsymbol{G}_y, \boldsymbol{G}_z$
  - Combined gradient with direction-dependent weights: $\boldsymbol{G} = [\alpha_x\boldsymbol{G}_x^\top, \alpha_y\boldsymbol{G}_y^\top, \alpha_z\boldsymbol{G}_z^\top]^\top$
- Second order derivatives:
  - Discrete Laplace operator on Yee cell

Regularization for Finite Elements

- Smoothness regularization:
  - Replace regularization term $R(\boldsymbol{m}) = \frac{1}{2}\|\boldsymbol{W}\boldsymbol{m}\|_2^2$ with $R(\boldsymbol{m}) = \int_\Omega \nabla\boldsymbol{m}\cdot\nabla\boldsymbol{m}\,\mathrm{d}\boldsymbol{x}$
  - Formulate equivalent problem with mixed formulation and dual variable
  - Discretize using Raviart-Thomas elements, yields $\boldsymbol{M}_{\text{RT}_0}$ and $\boldsymbol{D}$
  - Eliminate dual variable
  - Obtain $R(\boldsymbol{m}) = \boldsymbol{m}^\top\boldsymbol{D}\boldsymbol{M}_{\text{RT}_0}^{-1}\boldsymbol{D}^\top\boldsymbol{m}$ or simply $\boldsymbol{W} = \boldsymbol{M}_{\text{RT}_0}^{-1/2}\boldsymbol{D}^\top$

Inversion Result

- Regularization applied to $\approx$ 6200 parameters:
  - Gradients with doubled weight perpendicular to flight profiles
  - Second order derivatives with tenth of weight of gradients
- Obtained results (conductivity distribution) after 12 steps of Gauss-Newton:



Subset of an FD grid with conductivities obtained from an inversion for a layered background model with an embedded conductive block. The results shows an overall good agreement with the true model.

Summary and Outlook

- Done:
  - Fast forward solvers based on finite differences and finite elements
  - Efficient computation of sensitivities (both explicitly and as part of inversion)
  - Flexible regularization using a combination of available weightings
- Open tasks:
  - Integration of smoothness regularization for finite elements
  - Extensive model and (regularization) parameter studies
  - Inversion of more complicated models and field data (from AIDA SP2)

Siemon, B., Auken, E., & Christiansen, A. V. (2009). Laterally constrained inversion of helicopter-borne frequency-domain electromagnetic data. *J. Appl. Geophys.* 67, 259–268.

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